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DEVELOPMENT OF A PHYSICAL THEORY OF PLASTICITY

T. H. Lin and Sergio G. Ribeiro
University of California, Los Angeles, CA 90024, U.S.A.

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Abstract—A physical theory of plasticity is developed from the slip theory given by Lin and his associates. In this slip theory, a large region composed of innumerous identical cubic blocks of 27 differently oriented face-centered-cubic crystals is embedded in an infinite isotropic elastic medium. Loading is applied to the infinite medium. The stress and strain fields in the center block is calculated using the analogy between plastic strains and applied force. These fields satisfy the conditions of equilibrium and continuity of displacement throughout the block. The average stress and strain of this block is considered to represent the macro-stress–strain of the metal. The 27-crystal model gives some plastic shear strain under tensile loading that does not agree with test data. In this paper a 64-crystal model is used. These crystals are arranged to give symmetry about 3 orthogonal planes and same crystal orientations relative to the three perpendicular axes. With this model, the undesirable plastic shear strain is removed. Significant simplifications in the numerical calculations have been developed.

Calculations by this theory have been made on an aluminum alloy subject to three different ratios of incremental axial and torsional loadings after being compressed beyond the elastic range. These calculated results agree with experimental data in a thin wall cylinder, considerably better than the commonly used flow and deformation theories of plasticity. Lode parameters of strain were calculated for different Lode parameters of stress. These calculated results again agree considerably better than the commonly used mathematical theories of plasticity.

INTRODUCTION

The stress–strain relation of polycrystalline metals in the plastic range has been studied by many investigators. These studies may be divided into two classes: One is known as the mathematical theory of plasticity and the other the physical theory of plasticity[1]. A mathematical theory is mainly a representation of experimental data as a necessary extension of elasticity theory to furnish more realistic estimates of the load-carrying capacity of structures. Mathematical simplicity is essential to this representation so as to be readily applicable to design and analysis. As pointed out by Drucker[2], this type of plasticity theory is only a formalization of known experimental results and does not inquire deeply into the physical and chemical basis. This type of theory is generally started from hypothesis and assumptions of a phenomenological character based on certain experimental observations. The assumed phenomenological laws cannot claim generality and are apt to give a reliable approach only to a relatively limited class of real processes[3, 4]. On the other hand, a physical theory as one shown in the present paper, does attempt to explain why things happen the way they do, but may not embody mathematical simplicity. After a physical theory is developed, simplifications of the calculations are to be made. With the rapid advance of computers, it is hoped that the physical theory can also be applied to design and analysis of structures in the near future.

A polycrystalline metal is composed of numerous component crystals. The stress–strain relation of the metal is expected to depend on that of the component crystals. Hence, the polycrystal incremental stress–strain relation derived from that of component crystal gives a physical theory of plasticity and is expected to give a more realistic representation of the metals. In 1966, Lin and Ito[5] have developed a method to calculate the theoretical plastic stress–strain relation of a polycrystal from the stress–strain relation of single crystals. In this method, the plastic strain gradient is considered as an applied body force similar to Duhammel's Analogy for the analysis of thermal stresses. This method satisfies the condition of continuity of displacement and the condition of equilibrium. The component crystals in the polycrystal are assumed to have the same stress–strain relation of the single crystals. The size of single crystals in single crystal tests is much larger than that of the grains in polycrystals and it is known that the stress–strain relation of crystals varies with grain size. Hence there is error in assuming single crystal test data to be that of component crystals in a fine-grained polycrystal. Besides, single crystal test data are only available for pure aluminum and are not available for aluminum alloys. Since aluminum alloys are commonly used in engineering structures, it is our main interest to develop a physical theory applicable to these alloys. Hence in 1974, Lin *et al.*[6] developed a method to calculate the component crystal stress–strain relation from the experimental tensile stress–strain curve of the polycrystal. This approach is similar to the derivation of the characteristic shear function from the tensile stress–strain curve in the development of the first simplified slip theory of plasticity by Batdorf and Budiansky[7]. In this study[6], a large three-dimensional region in an infinite median is entirely filled with innumerous identical cubic blocks, each of which consists of 27 cube-shaped F.C.C. crystals of different orientations. The elastic constants of the crystals are taken to be isotropic and homogeneous and the same as the infinite medium. The 27 orientations were chosen such that the orientations of one specimen axis X_1 relative to the 27 sets of crystal axes on a standard stereographic projection of a cubic crystal are quite uniformly distributed in a unit stereographic triangle, while the specimen axes X_2, X_3 relative to these 27 sets of crystal axes are only approximately uniformly distributed in two other stereographic triangles. This 27-crystal model does not strictly satisfy the condition of orthotropy, hence some plastic *shear* strain occurs in tensile loading. This does not agree with test results of polycrystal metals.

INITIAL ISOTROPY OF METALS

Metals considered for the development of the plasticity theory are composed of crystals of random orientations. Let S_{ij} and E_{ij} denote the macroscopic stress and strain components respectively.

Under tensile loading S_{11}, plastic deformation generally satisfies $E^p_{22} = E^p_{33} = -\frac{1}{2}E^p_{11}$ $E^p_{12} = E^p_{23} = E^p_{31} = 0$. Similar conditions have been observed under uniaxial loadings S_{22} or S_{33}. Under a pure shear loading $S_{ij}(i \neq j)$ all plastic strain components except the corresponding E^p_{ij} vanish. These properties are referred to as the initial isotropy of polycrystalline metals. The 27-crystal model does not satisfy this property. In order to satisfy this property, a 64-crystal model[8] replacing the 27-crystal model has been developed.

The 64 crystals in a basic cubic block are divided into 8 groups, each of which occupies one octant of the block. The orientations and arrangement of crystals of one group are chosen so as to give mirror images of other groups with respect to three coordinate planes of the specimen axes. This gives three planes of symmetry and hence satisfied property of orthotropy. Among the 8 crystals in the first octant, crystals 1–8 in Fig. 1, one crystal has one slip system most favorable under S_{11} and another under S_{12}. This crystal is oriented such that the resolved shear stress in the most favorable slip system under S_{11} equals to that under loading S_{12} if the loading S_{12} equals $0.577S_{11}$. This would give Von Mises criterion for initial yielding. The orientation of this crystal can be chosen to give some other criterion of initial yielding (such as Tresca's). Another crystal is oriented to give a mirror image of the first with respect to the plane making 45° with both x_2 and x_3 axes. These give two crystals associated with S_{11}. Similarly there are two crystals associated with S_{22} and with S_{33}. These take six crystals. The remaining two crystals have their crystal axes coinciding with specimen axes x_1, x_2, x_3. These 8 crystals are positioned in the first octant in such a way as to give no preference to loadings S_{11}, S_{22} or S_{33}. By this way, the property of initial isotropy of polycrystals *referring to this set of axes* is fulfilled. This model is essential to represent the plastic deformation of randomly oriented polycrystalline metals.

Stress and strain in solids generally vary from point to point. Hill[9] in 1956 proposed to use the average stress and average strain to represent the macrostress and macrostrain of the polycrystal. Here we consider the average incremental stress $\Delta\bar{\tau}_{ij}$ vs the average incremental plastic strain $\Delta\bar{e}^p_{ij}$ of the interior *center* block to represent the macroscopic incremental stress ΔS_{ij} vs macroscopic incremental plastic strain ΔE^p_{ij} of the polycrystal. The surrounding blocks can deform plasticially. This causes the medium embedding the center block to be softer than a purely elastic medium.

NUMERICAL CALCULATIONS

Consider a point x' in the aggregate sliding in one or more slip systems. Let (y_i) be a set of rectangular coordinates with y_1 along the normal to the slip plane and y_2 along the slip direction of a slip system at the source point x', where slip has taken place. Denoting the plastic shear

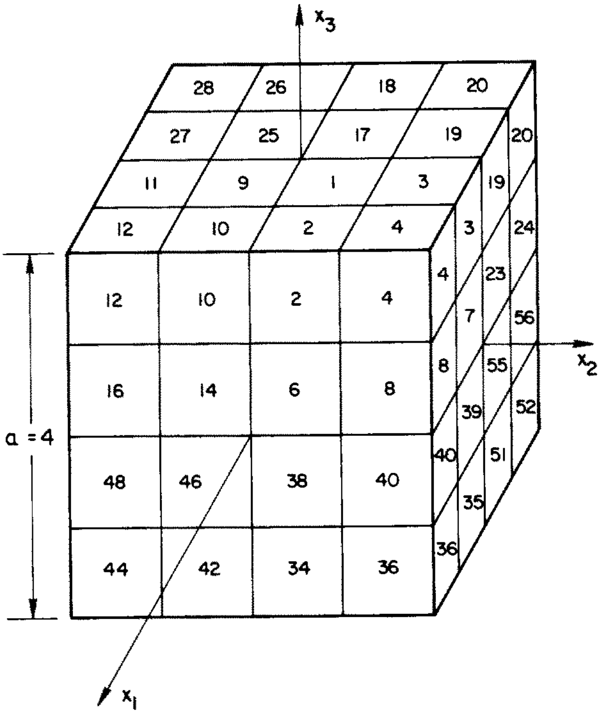

Fig. 1. Basic block of 64 cube-shaped crystals and the crystal numbers.

strain caused by slip in the nth slip system by $[\gamma_{y_1y_2}(x')]_n$, the equivalent body force components along x_k reference axes due to $[\gamma_{y_1y_2}(x')]_n$'s in N slip systems at this point x', is given as:

$$\bar{F}_k(x') = -2\mu \sum_{n=1}^{N} \left[\frac{\partial \gamma_{y_1y_2}(x')}{\partial x'_l} L_{kl}(x')\right]_n \tag{1}$$

where

$$L_{kl} = \frac{\partial x'_l}{y_2}\frac{\partial x'_k}{\partial y_1} + \frac{\partial x'_l}{\partial y}\frac{\partial x'_k}{\partial y_2}.$$

Since this large region composed of these 64-crystal cubic blocks and the surrounding infinite medium have the same isotropic elastic constants, hence the equivalent forces caused by plastic strain in this large region are then considered to apply in an infinite elastic medium. The displacement field in an infinite isotropic elastic medium caused by body force $\bar{F}_k(x')$ acting in a volume V' has been given by Kelvin[10, 11]. From this displacement field the stress field was obtained[5]. Let $\tau^S_{ij}(x)$ be the stress at x caused by the equivalent body force $\bar{F}(x')$. From Kelvin's solution, we have

$$\tau^S_{ij}(x) = \int_{v'} \phi_{ijk}(x, x')\bar{F}_k(x')\,\mathrm{d}v' \tag{2}$$

where

$$\phi_{ijk}(x, x') = \frac{-6\mu A(x_i - x'_i)(x_j - x'_j)(x_k - x'_k)}{r^5} + \frac{2\mu^2 A}{(\lambda+\mu)r^3}[\delta_{ij}(x_k - x'_k) - \delta_{ik}(x_j - x'_j) - \delta_{jk}(x_i - x'_i)],$$

$$r_2 = (x_i - x'_i)(x_i - x'_i) \quad \text{and} \quad A = \frac{\lambda+\mu}{8\pi\mu(\lambda+2\mu)}$$

$$\tau^S_{ij}(x) = -2\mu \int \phi_{ijk}(x, x')\left[\frac{\partial \gamma_{y_1y_2}}{\partial x'_l} L_{kl}(x')\right]\mathrm{d}v'. \tag{3}$$

To simplify the calculation, plastic strain in each crystal is assumed to be uniform and the stress at the centroid of each crystal is assumed to represent the stress over the entire crystal. By this assumption, plastic strain gradient in each crystal vanishes. However, across the crystal boundary, each crystal is considered to have the plastic strain dropped from this uniform value to zero, causing an equivalent uniform surface force on plane boundary surfaces of the crystal.

Equation (10) reduces to

$$\tau_{ij}^{S}(x) = 2\mu \oint \phi_{ijk}(x, x')\gamma_{y_1y_2}L_{kl}(x')\,\mathrm{d}S'_l$$

$$= 2\mu\gamma_{y_1y_2} \oint \phi_{ijk}(x, x')L_{kl}(x')\,\mathrm{d}S'_l \tag{4}$$

where $\mathrm{d}S'_l$ is the projected differential area of the boundary of the volume v' normal to x_l-axis. With ν_l denoting the cosine of the angle between the normal to the boundary and x_l-axis and $\mathrm{d}S'$, the differential boundary area

$$\mathrm{d}S_l = \nu_l\,\mathrm{d}S.$$

The residual stress field is then

$$\tau_{ij}^{R}(x) = \tau_{ij}^{S}(x) - 2\mu e_{ij}^{P}(x) \tag{5}$$

where the superscript P denotes the plastic strain.

Then the residual stress at the centroid of pth crystal $\tau_{ijqn}^{R}(p)$ due to slip on the nth slip system of the qth crystal $e^{P}(q, n)$ may be calculated and written as

$$\tau_{ijqn}^{R}(p) = a_{ijqn}(p)e^{P}(q, n) \tag{6}$$

where $a_{ijqn}(p)$ is the stress τ_{ij} in pth crystal caused by unit plastic shear strain in qth crystal caused by slip in the nth slip system. With a uniform stress τ_{ij}^{0} due to a uniform loading applied to the infinite medium, the resulting stress at the centroid of the pth crystal becomes

$$\tau_{ij}(p) = \tau_{ij}^{0} + a_{ijqn}(p)e^{P}(q, n). \tag{7}$$

Since the average incremental microstress $\Delta\bar{\tau}_{ij}$ over the cubic block of 64 crystals represents the incremental macrostress ΔS_{ij} of the metal

$$\Delta S_{ij} = \Delta\bar{\tau}_{ij} = \Delta\tau_{ij}^{0} + \bar{a}_{ij'qn}\Delta e^{P}(q, n) \tag{8}$$

where the bar on the top denotes the average value. From (7) and (8)

$$\Delta\tau_{ij}(p) = \Delta S_{ij} + (a_{ijqn} - \bar{a}_{ijqn})\Delta e^{P}(q, n)$$

$$= \Delta S_{ij} + b_{ijqn}\Delta e^{P}(q, n) \tag{9}$$

where b_{ijqn} denotes the terms in the parenthesis.

Let $\alpha_i(p, m)$ be the component along x_i-axis of the unit vector normal to the sliding plane of the mth slip system in pth crystal and $\beta_j(p, m)$, along x_j-axis of the unit vector along the sliding direction of the same slip system and

$$m_{ij}(p) = \alpha_i(p, m)\beta_j(p, m) + \alpha_j(p, m)\beta_i(p, m). \tag{10}$$

Then the incremental resolved shear stress in the mth slip system of the pth crystal is obtained by tensor transformation as

$$\Delta\tau(p, m) = m_{ij}(p)[\Delta S_{ij} + b_{ijqn}\Delta e^{P}(q, n)]$$

$$= m_{ij}(p)\Delta S_{ij} + C_{pmqn}\Delta e^{P}(q, n). \tag{11}$$

The coefficients C_{pmqn}'s are readily calculated. For those active slip systems, the incremental resolved shear shress $\Delta\tau(p, m)$ is equated to its incremental critical shear stress $\Delta\tau_c(p, m)$ and all the incremental plastic strains in different slip systems are calculated.

Previously for a given ΔS_{ij} (Lin *et al.* [2]) different values of τ_{ij}^0's were tried to obtain a $\Delta\bar{\tau}_{ij}$ close to the given ΔS_{ij}. Now with the relation (11), the given ΔS_{ij} is used as input and *no trial is needed*. This saves computation time to a large extent.

NUMERICAL RESULTS

Using the method described above, the critical shear stress vs the sum of slip of the component crystal of 14ST Al.Al., was calculated from the polycrystal tensile stress–strain curve and is shown in Fig. 2. The incremental plastic strains of this metal, under different ratios of incremental compression and torsional loadings after the material was stressed beyond yielding in compression were calculated. Budiansky *et al.* [12], have experimentally obtained incremental stress–strain data of this alloy under these loadings. It was found that as shown in Figs. 3–5, the present theory yields better representation of these experimental data than the flow and deformation theories.

Under radial loadings, from Von Mises criterion of loading, the ratio of any pair of principal shear stresses may be regarded as a definite function of the ratio of the corresponding shear

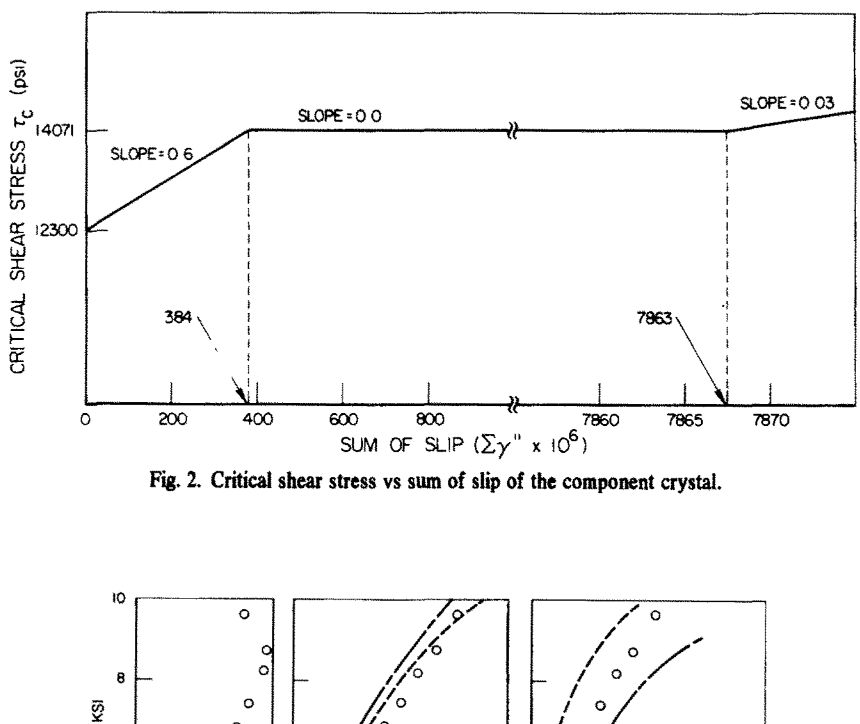

Fig. 2. Critical shear stress vs sum of slip of the component crystal.

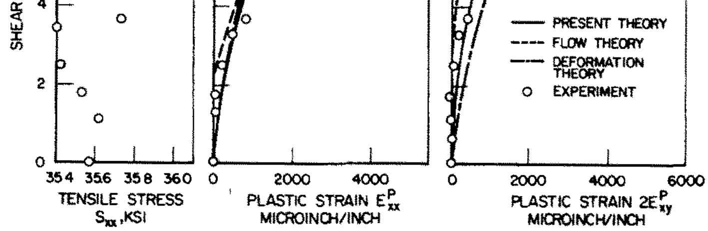

Fig. 3. Loading path and plastic strains for $(\Delta S_{xx}/\Delta S_{xy}) = 0.052$, $E^P_{yy} = E^P_{zz} = -\frac{1}{2}E^P_{xx}$, $E^P_{xz} = E^P_{yz} = 0$.

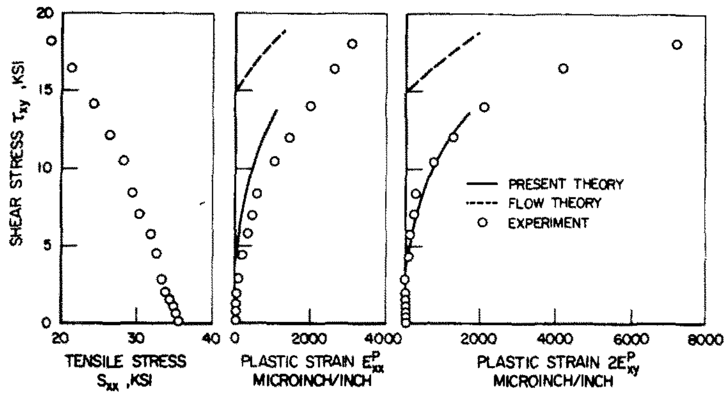

Fig. 4. Loading path and plastic strains for $(\Delta S_{xx}/\Delta S_{xy}) = -0.656$, $E^P_{yy} = E^P_{zz} = -\frac{1}{2}E^P_{xx}$, $E^P_{xz} = E^P_{yz} = 0$.

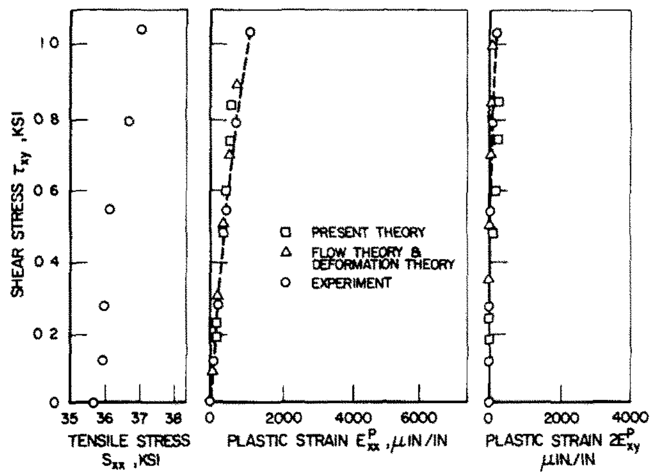

Fig. 5. Loading path and plastic strain for $(\Delta S_{xx}/\Delta S_{xy}) = 1.18$, $E^P_{yy} = E^P_{zz} = -\frac{1}{2}E^P_{xx}$, $E^P_{xz} = E^P_{yz} = 0$.

strains [13]. To check the validity of this relation, Lode proposed two parameters.

$$\mu = 2\left(\frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3}\right) - 1 \tag{12}$$

where σ_1, σ_2, σ_3 are the principal stresses, and

$$\nu = 2\left(\frac{e_2 - e_3}{e_1 - e_3}\right) - 1 \tag{13}$$

where e_1, e_2, e_3 are the principal strains. Von Mises' criterion predicts $\mu = \nu$. However, experimental data consistently show that numerically ν is less than μ [13, 14]. Calculations based in the presently developed theory as shown in Fig. 6 also show this consistent deviation of ν from μ. Again it is seen that the present theory does represent the actual deformation better than the presently used plasticity theories.

CONCLUSIONS AND DISCUSSIONS

(1) This theory gives a method to calculate the incremental stress–strain relation of aluminum and its alloys under combined loadings either radial or non-radial from the experimental uniaxial stress–strain curves of the metal.

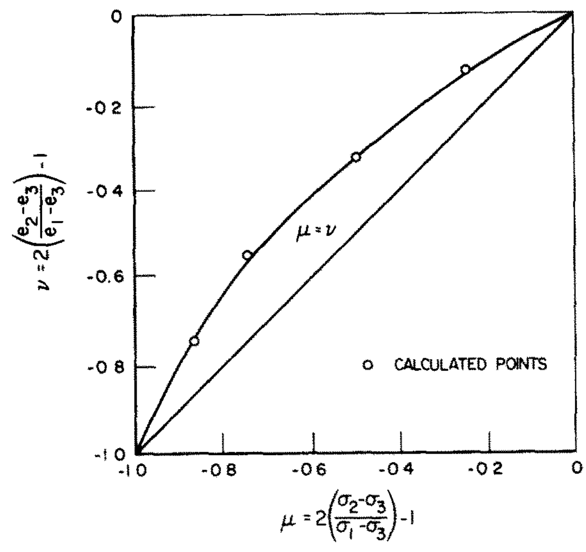

Fig. 6. Lode's parameters calculated from present theory at octahedral stress ≅ 37,000 psi.

(2) This theory satisfies the condition of equilibrium, continuity of displacement and the dependency of slip on the resolved shear stress.

(3) In this theory, the resolved shear stress vs slip relation is calculated from the polycrystal uniaxial stress–strain curves. Hence the grain size effect on this relation is automatically included.

(4) This polycrystal model referring to a set of rectangular axes satisfies the condition of initial isotropy of polycrystalline metals, i.e. under axial loadings S_{11}, plastic strain $E^P_{22} = E^P_{33} = -\frac{1}{2}E^P_{11}$ and $E^P_{12} = E^P_{23} = E^P_{31} = 0$, same stress–strain relations under axial loadings S_{22} or S_{33} and under shear loading S_{ij}, $i \neq j$, all plastic strain components except the corresponding E^P_{ij} vanish.

(5) The incremental stress–strain curves of metals subject to three different ratios of compressive and shear stresses after being compressed beyond the elastic range, calculated by the present theory agree considerably better with experimental results than the commonly used mathematical theories of plasticity (flow and deformation theories).

(6) For different Lode parameters for stress μ, the corresponding Lode parameters for strain ν were calculated by the present theory. The variation of ν with μ given by the present theory agree much better with experimental results than the currently used mathematical theories of plasticity.

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